Interaction of Shock and Rare-
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Motion.

by

R. Courant and K. Friedrichs
New York University

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Progress Report on "Interaction of Shock and Rarefaction Waves in One-Dimensional Motion"

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Interaction of Shock and Rarefaction Waves

in One-Dimensional Motion

The reflection of a straight shock wave on a rigid wall, the head-on collision of two straight shocks, and the collision of two shocks in the same direction overtaking each other, was investigated from a mathematical point of view in particular by von Neumann (see Report No. 1140, O. S. R. D. and three memoranda referred to therein). From such collisions not only shocks but also rarefaction waves and "contact discontinuities" may result(*). To obtain a complete understanding of the possible phenomena involving discontinuities in a linear motion of a compressible gas, it seems therefore natural to consider shocks, rarefaction waves and contact discontinuities on the same footing, i.e., to study the effect of the interaction of <u>any two</u> or more of them. The present memorandum is concerned with the systematic study of such possible

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(*) A "contact discontinuity" or "contact surface" occurs when two layers of gas in contact have the same pressure and velocity, but different densities and (naturally) different entropies and temperatures. The importance of these contact discontinuities for the understanding of the phenomena was clearly recognized by von Neumann.

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interactions in one-dimensional motions of compressible gases or fluids.

Phenomena of this kind may be produced by letting two pistons act at both ends of a tube filled with gas. The effect may consist in shocks or rarefaction waves moving toward each other or following each other, or, in case of "contact surfaces", shocks or rarefaction waves passing from one layer into another. As soon as two waves or shocks meet, or one crosses a contact layer, a rather complicated gas-dynamical process will begin. However, in many cases this complicated process of "penetration", either immediately (*) or after some time, results in a much simpler "terminal" state, characterized by two (either shock or rarefaction) waves which move steadily away from each other and are separated by a region of constant pressure and particle velocity. In this intermediate region new contact discontinuities may occur. There are cases in which such a simple description of the final

(*) Such immediate separation takes place if two shocks meet or a shock crosses a contact discontinuity.

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state is not adequate. However, in the present investigation the <u>assumption</u> is expressly made that a <u>simple terminal state</u> as described will result. Under this assumption it is possible to find the terminal state without analyzing the process of penetration, i.e. without integrating the differential equations of gas dynamics for this process.

In some of the interactions considered our assumption implies a new phenomenon: simple contact discontinuities will not occur but rather "<u>contact zones</u>"; i.e., columns of constant length, (moving with constant velocity through the tube) over which density and entropy vary continuously. In these cases our basic assumption of a simple terminal state can serve only as an approximation, which has to be justified and improved by a more complete solution of the differential equations (*).

It must be emphasized that the study of one-dimensional motion can be considered only as a preliminary attempt at understanding the greater

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(*) This has been carried out in a typical case. (See page 3[illegible]).

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variety of analogous phenomena in three dimensions. The results of the present memorandum must furthermore be confronted more throughly with experimental evidence.

The investigation will be based on a convenient form of the Rankine - Hugoniot conditions for a shock and in a parallel way conditions of transition across a rarefaction wave. Furthermore, the occurrence of contact discontinuities makes it advisable to introduce as basic variables not, as is customary, pressure and density, but rather the pressure p and the particle velocity u, thereby avoiding complications by changes of density and entropy. The representation of possible states and transitions in a (u,p)- diagram makes it possible to analyze the phenomenon by graphical methods (which of course have to be supplemented by numerical procedure to obtain precision, not only qualitative results).

In this report we assume an ideal gas. Yet the method can be applied to substances with a different equation of state.

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1. Shocks, Rarefaction Waves, and Contact Discontinuities

The motion shall take place along the x-axis. Throughout velocities are counted positive if in the direction of the positive x-axis. Particle velocities are denoted by u, shock velocities by U. The pressure is denoted by p, the density by ρ, the specific volume by $\tau = \rho^{-1}$. We assume that a point x_0 (or an interval) separates two sections of a column of gas each having constant velocity, pressure, and density. The regions $x < x_0$, $x > x_0$, or generally the regions of constant p and u on the left and right are characterized by the subscripts $_l$ ("ell") and $_r$ respectively.

Suppose first a shock transition takes place at the point x_0. We call it a "forward shock", $\underset{\rightarrow}{S}$, if the particles cross it from r to l, a "backward shock", $\underset{\leftarrow}{S}$, if the particles cross it from l to r. According to the principle that all shocks are compression shocks, one has the inequalities

$$(1.1) \qquad p_l > p_r, \quad \rho_l > \rho_r, \quad \text{for } \underset{\rightarrow}{S},$$

$$p_l < p_r, \quad \rho_l < \rho_r, \quad \text{for } \underset{\leftarrow}{S},$$

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while for the velocities one has in both cases

(1.2) $u_l > u_r$ for $\underset{\rightarrow}{S}$ and $\underset{\leftarrow}{S}$

(a remark quite useful for the later discussion.)

A continuous transition from a region of constant state into another region of constant state can be affected only by a "<u>simple wave</u>", i.e., a zone in the tube for which one set of characteristics in the (x,t)-plane is straight. If those characteristics diverge as time goes on or what is equivalent if the particles move apart from each other the wave is called "<u>rarefaction wave</u>".(*). We speak of a forward or a backward rarefaction wave $\underset{\rightarrow}{R}$ or $\underset{\leftarrow}{R}$ according to whether the particles pass through the wave zone from the right to the left or from the left to the right respectively. For pressure and density on the two sides one has

(1.3) $p_l < p_r$, $\rho_l < \rho_r$ for $\underset{\rightarrow}{R}$

$p_l > p_r$, $\rho_l > \rho_r$ for $\underset{\leftarrow}{R}$

(*) It need not necessarily originate in a discontinuity but may come, for example, from a decelerated piston. Other simple waves containing contracting zones will lead to shocks eventually and are called <u>compression</u> or <u>contraction waves</u>; they may come from an accelerated piston.

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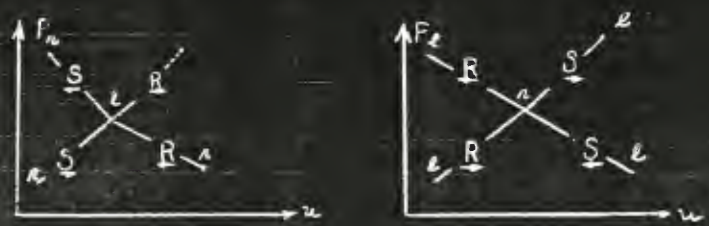
while for the velocities one has

(1.4) $u_l < u_r$ for $\underset{\rightarrow}{R}$ and $\underset{\leftarrow}{R}$ i.e., in both cases.

These relations correspond to relations (1.1) and (1.2) for shocks.

Summarizing we can represent the relative position of possible states r connected with a given state l, or states l connected with a given state r in the following diagrams respectively.



One observes that the quotient

$$(p_r - p_l)/(u_r - u_l)$$

is always positive for forward waves and negative for backward waves.

The characteristics (green color) and particle paths (blue) are shown in the accompanying (x,t) diagrams for both shocks (red) and rarefaction zones (the latter swept by a family of straight characteristics).

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Finally, "contact surfaces" should be distinguished, depending upon which side of the discontinuity has the greater density. A "contact discontinuity" will be called "T <" if $\rho_l < \rho_r$ and "T >" if $\rho_l > \rho_r$. The equality of pressures on both sides of the discontinuity will immediately lead to inequalities among quantities that are determined by pressure and density jointly such as absolute temperature θ, entropy H, and speed of sound c, i.e.,

$$(1.5)\quad (T<)\ \rho_l < \rho_r: \quad c_l > c_r,\ \theta_l > \theta_r,\ H_l > H_r$$

$$(T>)\ \rho_l > \rho_r: \quad c_l < c_r,\ \theta_l < \theta_r,\ H_l < H_r$$

It may be noted that the two states separated by a contact surface are represented by the same points in the (u,p) diagram.

2. Transition Relations for Shock and Rarefaction Waves

We assume an ideal gas with the adiabatic exponent γ such that $p\rho^{-\gamma} = p_0\rho_0^{-\gamma}$ is the equation of state for adiabatic changes, ($\gamma = 1.4$ for the usual case of a diatomic gas, $\gamma = \frac{5}{3}$ for monatomic gases and $\gamma = \frac{4}{3}$ for polyatomic gases). We define

(2.1) $$\mu = \frac{\gamma + 1}{\gamma - 1} \qquad (\mu = 6 \text{ for diatomic gases}).$$

A. Relations for shocks.

If the regions l and r are separated by a shock with velocity U, we obtain from the Rankine-Hugoniot condition the relations

(2.2) $$\rho_r / \rho_l = \tau_l / \tau_r = h(p_r/p_l) \quad \text{where}$$

(2.3) $$h(x) = \frac{\mu x + 1}{x + \mu}$$

With the general notation $[f(x)]$ for the jump $f(x_l) - f(x_r)$ of a function $f(x)$ if x_l and x_r approach the point x_0 of discontinuity from the left and from the right respectively, we have further

(2.4) $$[\tau][p] = -[u]^2$$

(2.5) $$U_{rl} = u_r - \tau_r \frac{[u]}{[\tau]} = u_l - \tau_l \frac{[u]}{[\tau]}$$

again as a consequence of the Rankine-Hugoniot conditions.

Now we express $[\tau]$ in terms of p_r, p_l using (2.2), then, in view of (1.2), we obtain from (2.4)

$$(2.6)\quad [u] = \pm[p]\sqrt{\frac{(\mu - 1)\tau_r}{\mu p_r + p_l}} = \pm[p]\sqrt{\frac{(\mu - 1)\tau_l}{\mu p_l + p_r}}$$

for $\vec{S}$ and $\overleftarrow{S}$.

It is convenient to introduce the function

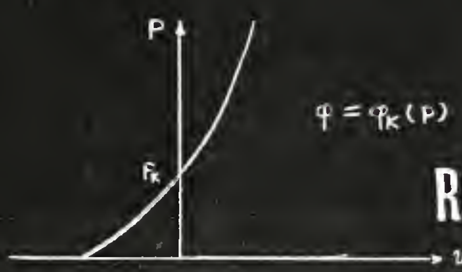
$$(2.7).\qquad \varphi_k(p) = (p - p_k)\sqrt{\frac{(\mu - 1)\tau_k}{\mu p + p_k}}$$

which depends upon two positive parameters p_k and ρ_k assigned to the state k. $\varphi_k(p)$ represents the difference of the normal particle velocities across a shock line as a function of the pressure on one side of the line when the pressure p_k and the density ρ_k is given on the opposite side. Consequently

$$(2.8)\qquad \varphi_l(p_r) = -\varphi_r(p_l)\ ,$$

if the regions r and l are connected by a shock. More generally, in virtue of (1.2)

$$(2.9)\qquad u_l - u_r = |\varphi_l(p_r)| = |\varphi_r(p_l)|$$



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Obviously, $\varphi_k(p)$ is a monotone increasing function of p and its derivative $\varphi_k'(p)$ is a monotone decreasing function of p, symbolically

$$\varphi_k(p) \uparrow \infty, \quad \text{as} \quad p \uparrow \infty;$$
$$\varphi_k'(p) \downarrow 0, \quad \text{as} \quad p \uparrow \infty.$$

Furthermore it will be useful to make the following simple remark concerning the dependence of $\varphi_k(p)$ on k:

<u>Remark</u> 1. If $\rho_k < \rho_h$ and $p_k \leq p_h$, then

$$\varphi_k(p) > \varphi_h(p) \quad \text{for} \quad p \geq p_k .$$

The curves

(2.10) $\quad u = u_k + \left|\varphi_k(p)\right| \quad$ and $\quad u = u_k - \left|\varphi_k(p)\right|$

will be called the $\overrightarrow{S}$- and $\overleftarrow{S}$-curves through k or the curves $\overrightarrow{S}_k$ and $\overleftarrow{S}_k$ respectively. (In diagrams we omit the arrows since the positive or negative slope of an S-curve is sufficient to indicate that it refers to a forward or backward shock respectively). A graphical representation of the possible states $u = u_r$, $p = p_r$ if state 1 is prescribed (or, if state r is prescribed, the possible states $u = u_1$, $p = p_1$) is shown by the diagram.

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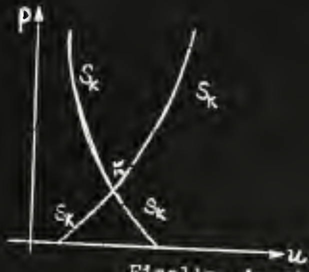

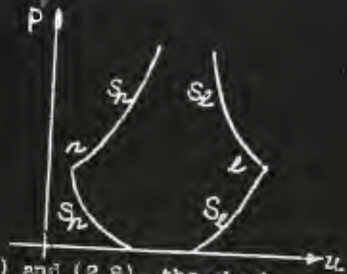

Finally, in view of (2.5) and (2.8), the shock velocity U_{rl} is given by

$$(2.11) \qquad U_{rl} = u_r + \tau_r \frac{[p]}{[u]} = u_l + \tau_l \frac{[p]}{[u]} ,$$

where $[p] = p_l - p_r$ and $[u] = u_l - u_r$. Thus if $S = \overrightarrow{S}$, then $\frac{[p]}{[u]} > 0$ and hence $U_{rl} > u_l > u_r$. On the other hand, if $S = \overleftarrow{S}$, then $U_{rl} < u_l < u_r$, since $\frac{[p]}{[u]} < 0$.

5. Relations Involving Rarefaction Waves.

For particles moving across a rarefaction wave we have adiabatic changes of state and hence the relation

$$(2.12) \qquad \rho_l / \rho_r = (p_l/p_r)^{1/\gamma} .$$

It is decisive that for rarefaction waves an analogue to (2.6) follows by integration of the differential equation of motion. With the notation $\eta = \frac{1}{2}(1 - \gamma^{-1}) = \frac{\gamma - 1}{2\gamma}$ the change $[u] = u_l - u_r$ of the

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velocity u across a simple wave may be expressed by

$$(2.13)\quad [u] = \pm \sqrt{\mu^2 - 1}\ (\tau_r^{\frac{1}{2}}\, p_r^{\frac{1}{2}-\eta})\ (p_1^{\eta} - p_r^{\eta}),$$

the plus sign referring to $\overrightarrow{R}$ and the minus sign to $\overleftarrow{R}$.(*)
We introduce the function

$$(2.14)\quad \psi_k(p) = \sqrt{\mu^2 - 1}\ (\tau_k^{\frac{1}{2}}\, p_k^{\frac{1}{2}-\eta})\ (p^{\eta} - p_k^{\eta}),$$

which also depends upon two positive parameters p_k and $\tau_k^{\frac{1}{2}}\, p_k^{\frac{1}{2}-\eta}$ (the latter is essentially the entropy); then, corresponding to (2.9) we have

$$(2.15)\quad [u] = u_1 - u_r = -|\psi_r(p_1)| = -|\psi_1(p_r)|.$$

Finally analogous to (2.8), there exists the relation

$$(2.16)\qquad \psi_r(p_1) = -\psi_1(p_r),$$

if 1 and r are connected by a rarefaction wave.

Some useful properties of the function $\psi_k(p)$ are

$$\psi_k(p) \uparrow \infty \quad , \text{ as } p \uparrow \infty$$
$$\psi_k'(p) \downarrow 0 \quad , \text{ as } p \uparrow \infty$$

and $\psi_k(p)$ approaches the ψ-axis tangentially at

(*) Across a rarefaction wave $\overrightarrow{R}$ or $\overleftarrow{R}$, $u \mp (\mu - 1)c =$ const. (cf. Panel Memorandum) or $u \mp \sqrt{(\mu^2 - 1)\tau p} =$ const. In view of (2.12) we have $\tau_r^{\frac{1}{2}}\, p_r^{\frac{1}{2}-\eta} = \tau_1^{\frac{1}{2}}\, p_1^{\frac{1}{2}-\eta}$; hence (2.13) follows.

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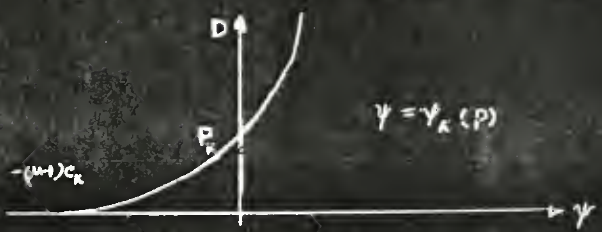
$$(2.17) \quad \psi = -\sqrt{(\mu^2 - 1)\,\tau_k\, p_k} = -(\mu - 1)\, c_k$$



In addition, we also have

Remark 2. If $p_k < p_h$ and $\tau_k\, p_k^{1/\gamma} \leq \tau_h\, p_h^{1/\gamma}$, then

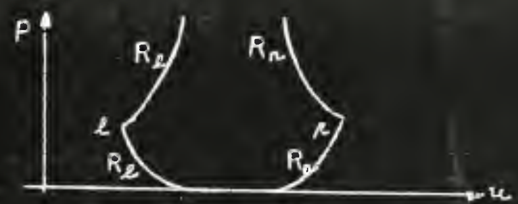
$$\psi_k(p) > \psi_h(p) \quad \text{for} \quad p \leq p_h .$$

As before, the curves

$$(2.18) \qquad u = u_k + |\psi_k(p)| \quad \text{and} \quad u = u_k - |\psi_k(p)|$$

will be called $\underset{\rightarrow}{R}$- and $\underset{\leftarrow}{R}$- curves through k or the curves $\underset{\rightarrow}{R}_k$ and $\underset{\leftarrow}{R}_k$ respectively. (In diagrams we omit the arrows since the positive or negative slope of the R-curve is sufficient to indicate that it refers to a forward or backward rarefaction wave respectively).

Graphically (*) the possible states $u = u_r$, $p = p_r$ if state l is prescribed (or, if state r is prescribed, the possible states $u = u_l$, $p = p_l$) are shown by the diagram.



Finally we note some relations between the functions $\varphi_k(p)$ and $\psi_k(p)$ (demonstrated in Appendix A4)

(2.19) $$\varphi'_k(p_k) = \psi'_k(p_k) , \quad \text{and}$$

(2.20) $$\varphi''_k(p_k) = \psi''_k(p_k) .$$

C. Contact Discontinuities.

For a contact discontinuity, there are no transition relations between the values of velocity and pressure, since across such a discontinuity these quantities remain continuous while only the density or entropy suffers the discontinuity.

(*) Branches not shown here would correspond to contraction waves.

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Riemann's Problem

In his classical paper on "Luftwellen von endlicher Schwingungsweite" Riemann discussed a problem closely related to our topic: At the time $t = 0$ an infinite "linear gas" column along the x-axis is divided by the point $x = 0$ into two constant states l and r; It is required to determine the subsequent state of the gas. Riemann showed that the initial discontinuity may resolve in either two shocks moving apart or two rarefaction waves (of the special character with characteristics meeting at a singular point) or one shock and one rarefaction wave. However, Riemann's solution is not complete, in fact, in Riemann's theory no contact surfaces are postulated since only two shock conditions are used which can be satisfied without introducing lines T in the (x,t) plane.

We shall now give a complete solution of Riemann's problem by a method that will likewise be applicable to all our problems of interaction. The solution consists in showing that we can always determine uniquely states l, m_*, r following the initial situation, such that the intermediate constant state m_* is connected with l by a backward wave, with r by a forward wave, each of which, according to circumstances, may be either a shock

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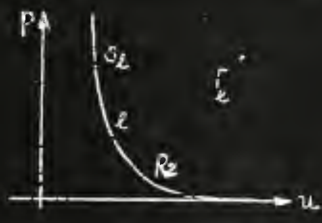
or a rarefaction wave. (Under certain circumstances no intermediate state will result, see below p. 19).

To reproduce such a solution, we realize what possible backward waves can connect a state m_s with l and what forward waves can connect m_s with r. (In our general scheme m_s will first play the role of r, then the role of l). Now, from our previous geometrical consideration it is clear, that in a (u,p) diagram all points representing such states which can be connected with l by a backward wave $\underleftarrow{R}$ or $\underleftarrow{S}$, will be on a "left transition curve" Γ_l, consisting of an upper branch of an $\underleftarrow{S}$ curve and a lower branch of an $\underleftarrow{R}$ curve. Similarly we have a right transition curve Γ_r connecting the point r with other points representing those states, for which the transition to r is effected by a forward wave $\underrightarrow{R}$ or $\underrightarrow{S}$.



Now we simply mark in the (u,p) plane the two points l and r corresponding to the prescribed initial states, and draw the two curves Γ_l and Γ_r

in the diagrams.

No matter where the two points l and r lie, as the basic properties of Γ_l and Γ_r curves show, there will always be (*) one and only one point of intersection m_* except when

$$(3.1) \qquad u_r - (\mu - 1)\, c_r > u_l + (\mu - 1)\, c_l$$

in which case the curves Γ_l and Γ_r both reach the u-axis tangentially without intersecting.

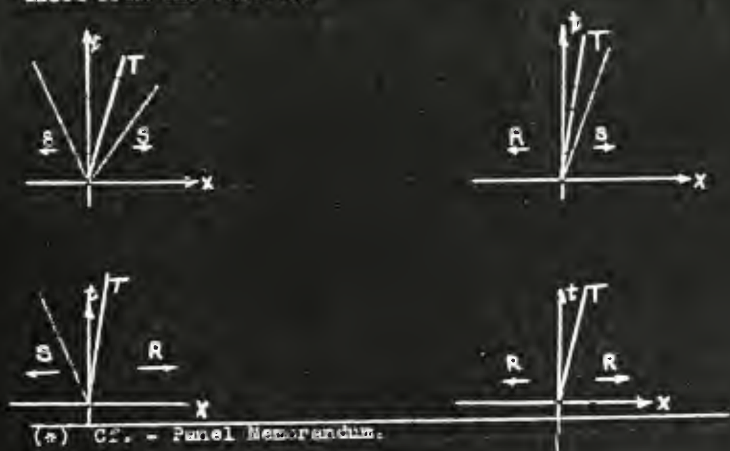
According to the positions of l and m and to the values of the parameter ρ_l, p_l and ρ_r, p_r the points of intersection m_* may lie within the S or R branch on either Γ_l or Γ_r and thus four possibilities arise. (In the exceptional case when relation (3.1) holds no intermediate state m_* will result, as the process of penetration continues indefinitely. Nevertheless we may

(*) The graphical construction of the intersection (u_*,p_*) can easily be refined or replaced by numerical calculation.

read the velocities u of both ends of the penetration zone from the points of tangency of Γ on the u-axis $[p = 0]$). (*)

If there is a regular intersection m_*, and if at least one of the two resulting waves is a shock, we shall find in general [by (2.2)] two different values ρ_{l*} and ρ_{r*} for ρ_* in m_* according to whether we determine ρ_* by a transition from the left or from the right. This must be interpreted by the occurrence of a contact discontinuity T, whose motion (or direction in the (x,t)-plane) is given by the velocity u_*.

The advantage of considering p and u as independent variables is that such contact discontinuities are obtained as a by-product. (They need not be introduced from the outset).



(*) Cf. - Panel Memorandum.

4. Interactions.

The basic problem of interactions is the following: given three constant states l, m, r separated by two waves R, S or by T, the waves should move in such a way that they intersect, destroying m.

We assume that after a period of penetration a terminal state l, m_*, r results, l, r being the same as before, m_* being the new middle zone of constant pressure p_* and velocity u_*, while the density ρ_* is permitted to vary discontinuously or even continuously from particle to particle in m_*. It is further assumed that m_* is separated from l by a backward wave, and from state r by a forward wave. The problem solved in this memorandum is to determine the state m_* and the two waves connecting m_* with l and r. As to the penetration process, its possibilities will be discussed only in a qualitative manner.

The initial situation is to be characterized by five independent quantities. In case the three states are separated by shock or rarefaction waves we choose the three pressures p_l, p_m, p_r and in addition the velocity u and the density ρ for one of the states.(*)

(*) It may be mentioned that the problem is essentially characterized by the two pressure ratios p_r/p_m and p_l/p_m. For a change in velocity of one state only implies a translation. Furthermore, a change in the pressure and density of one state, keeping the pressure ratios fixed, implies only a corresponding change in the scales of pressure, density, and velocity.

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Then all other quantities ρ_1 and u_1 are immediately determined by the transition conditions. The velocities of the separating shocks or the velocities of both ends of rarefaction waves are also determined. The five quantities are restricted only by the condition that the initial state should lead to a collision, i.e., that region m should shrink to zero.

In the case of a contact discontinuity not p but ρ is to be prescribed on both sides of it.

In any case we know the states l and r. The terminal middle state m_* is then determined both from the states l and r and from the condition that the transition lm_* be backward, and rm_* be forward. The situation is exactly the same as in Riemann's problem. Consequently the state m_* is uniquely determined as in Section 3.

The difference to Riemann's problem consists in the fact that in the present problem the regions l and r need not be separated by a point during the period of penetration, (e.g., they are not if two rarefaction waves collide). On the other hand the states l and r in the present problem cannot be chosen arbitrarily but are connected with each other through the original middle state m. (Two independent states would be characterized by 6 quantities, while here only 5 are prescribed).

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The type of the resulting new waves depends as in Riemann's problem on the relative position of the points l, r in the (u,p)-diagram together with ρ_l, ρ_r. Inasmuch as these two states are dependent on the state m in the present problem, the type of these new waves will depend on the initial situation i.e., on the states l, m, r and the type of original waves.

We shall now proceed to describe the various special cases.

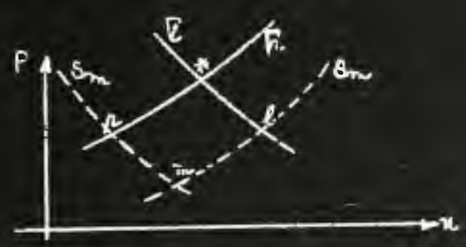
5. Head on Collisions.

5 A. Collision of Two Shocks.

Assume a $\overrightarrow{S}$ from region l and a $\overleftarrow{S}$ from r move toward each other; in this case

(5.1) $p_m < p_l$, and $p_m < p_r$.

It is easily seen that this condition is the only restriction on the intial data in the case of two shock waves moving toward each other. The relative position of the states l, m, r is shown in the diagram.



To fix the ideas we assume further

(5.2) $p_l \geq p_r$.

The transition curves Γ_l through l and Γ_r through r intersect in the upper branches of both and that therefore the two new transitions are shocks. This may be expressed symbolically by

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$$\underset{\rightarrow}{S} \quad \underset{\leftarrow}{S} \dashrightarrow \underset{\leftarrow}{S} \quad T \quad \underset{\rightarrow}{S},$$

where the contact discontinuity occuring in the middle region m_* is indicated by T.

To see that this construction is possible note that a point of intersection of $\bar{1}$ and r would be on the upper or S-branch of Γ_r since by assumption $p_1 \geq p_r$ and $u_1 > u_r$ (cf. (1.2)). Further from remark 1 in section 2 one sees that the slopes du/dp of the S-curve through m is larger than that of the S-curve through r provided $p > p_r$. This makes it clear that the S-curve through r passes the point 1 at its <u>left</u> side whence the statement immediately follows.

After u_* and p_* are determined one obtains the shock velocities U_{1*} and U_{r*} and the two densities ρ_{1*} and ρ_{r*}.

For the resulting quantities the following inequalities can easily be proved (see Appendix A 2)

$$u_* > u_m \quad \text{if} \quad p_1 > p_r. \tag{5.3}$$

This means that the stronger shock (from the left) will impart to the middle zone a velocity in its own direction, which is plausible. Another relation is

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(5.4) $\qquad p_* < p_l p_r / p_m \qquad$ or

$$p_*/p_l < p_r/p_m\,, \quad p_*/p_r < p_l/p_m;$$

this can be interpreted that shocks after penetrating each other have weakened each other. It may be mentioned that in case of a symmetrical clash, $p_l = p_r$, the relation p_*/p_r, is given by

(5.5) $$p_*/p_l = \frac{(\mu+2)p_l - p_m}{\mu p_m + p_l} \qquad \text{if } p_r = p_l\,;$$

it then is clear from the diagram that

(5.6) $$p_*/p_l < \frac{(\mu+2)p_l - p_m}{\mu p_m + p_l} \qquad \text{if } p_r < p_l$$

This estimate (5.6) is better than (5.4) if p_r/p_m is large. Indeed (5.6) implies the upper limit

(5.7) $\qquad p_*/p_l < (\mu+2) \qquad$ (= 8 for diatomic gases)

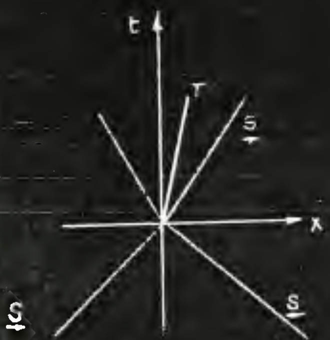
which is approached if $p_l/p_m \to \infty$.

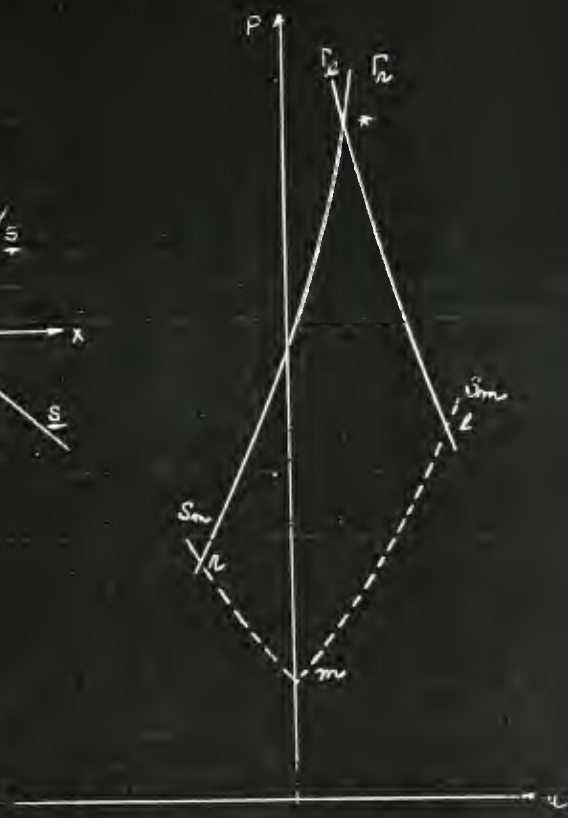
A further relation is

(5.8) $\qquad \rho_{l*} < \rho_{r*} \;$ if $p_l < p_r$.

This relation shows that in the middle region the shock reverses the relative magnitudes of the densities, a result which is intuitively not so obvious. In particular, it shows that the two densities ρ_{l*}, ρ_{r*} are always different and that there is always a

contact discontinuity in m_2 with the only exception that the two exterior states were equal, that is that the initial situation was symmetrical.





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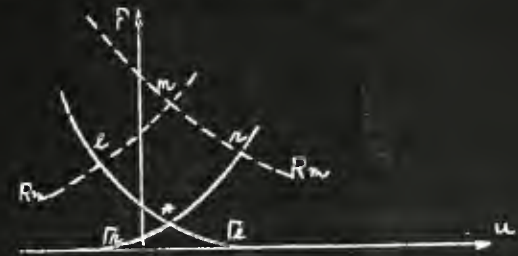
5 E. Collision of Two Rarefaction Waves.

If two R-waves are facing each other originally we have, in view of (1.3),

(5.9) $$p_m > p_l\,,\ p_r\,,$$

a condition which is also sufficient for two R-waves to move toward each other. From (1.4) we also have $u_l < u_r$. Therefore the relative position of the states l, m, r is indicated in the (u,p) diagram



The transition curves Γ_l and Γ_r through l, r meet in the lower R-branches (unless the states l and r satisfy the condition of cavitation (3.1)). This follows immediately from Remark 2 in section 2 (p. 15), by virtue of the fact that l, m, r have the same entropy. The result is that in the terminal state two rarefaction waves

$\underset{\leftarrow}{R}$, $\underset{\rightarrow}{R}$ move away from each other. Obviously there will be no contact discontinuity in the region m_*.

Symbolism

$$\underset{\rightarrow}{R} \quad \underset{\leftarrow}{R} \longrightarrow \underset{\leftarrow}{R} \quad \underset{\rightarrow}{R} \, .$$

The terminal state will not follow the initial state abruptly, but only after a period of penetration. The phenomenon is indicated in the (x,t)-diagram. That this picture represents correctly the solution of the underlying gas-dynamical differential-equation problem can be proved on the basis of the theory of characteristics.

From

$$\Gamma_m(p_l) - \Gamma_l(p_*) = - \Gamma_m(p_r) + \Gamma_r(p_*)$$

we derive, in view of constant entropy

(5.10) $$p_* = p_l \quad , \quad p_r = p_m$$

the right member being positive unless the condition (3.1) is satisfied, as is easily verified.

Clearly, since the entropy is constant,

$$(5.11) \qquad \rho_* = \rho_m \, (p_*/p_m)^{1/\gamma}$$

Also

$$(5.12) \qquad u_* = u_m + \sqrt{\mu^2 - 1} \, (\tau^{\frac{1}{2}} p^{\frac{1}{2}-\eta}) \left\{ p_r^{\eta} - p_1^{\eta} \right\}$$

hence

$$(5.13) \qquad u_* < u_m \quad \text{if} \quad p_1 < p_r \, .$$

For the sound velocities of the ends of the rarefaction waves one finds

$$(5.14) \qquad \begin{aligned} u_r + c_r &> u_m + c_m \\ u_* + c_* &> u_1 + c_1 \end{aligned}$$

and correspondingly

$$\begin{aligned} u_1 - c_1 &< u_m - c_m \\ u_* - c_* &< u_r - c_r \, , \end{aligned}$$

they follow from the theory of rarefaction waves (see separate memorandum).

RESTRICTED

5 C. Collision of Shock and Rarefaction Wave.

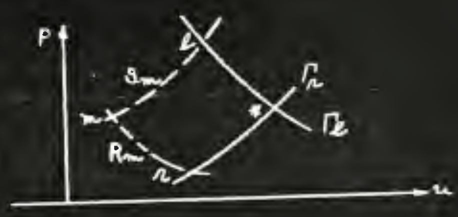
We consider the case that a R-wave $\overleftarrow{R}$ and a shock $\overrightarrow{S}$ meet. The relative positions of the states l, m, r is as in the diagram, the only condition on p is

(5.15) $p_l > p_m > p_r$.

The transition curve Γ_l through l does not pass between m and r, it crosses the line $p = p_r$ to the right of r, this follows from Remark 2, section 2 (p. 15), since

$$p_m < p_l , \quad \tau_m p_m^{1/\gamma} < \tau_l p_l^{1/\gamma}$$

since shock increases entropy.



The transition curve Γ_r through r does not pass between m and l, it crosses the line $p = p_l$ to the right of l, this follows from Remark 1 in section 2, since

$$p_r < p_m, \quad \rho_r < \rho_m .$$

Consequently the point * is on the lower branch of

Γ_l and on the upper branch of Γ_r i.e., the resulting waves are a $\overleftarrow{R}$ and a $\overrightarrow{S}$.

This result applies to the terminal state. The actual process of penetration may be described as follows: The shock will enter the zone of rarefaction, thereby continuously changing velocity and intensity, until it reaches the constant terminal state. Those particles which have crossed the shock during the pentration process will have suffered different changes in density and entropy and therefore will constitute a "contact zone", i.e., a zone in which the density does not vary discontinuously across a contact surface but continuously.

Our assumption of a constant pressure and velocity in the terminal state implies that all particles in this zone move with the same velocity, a situation which is compatible with the equations of motion of gas dynamics if we take into account the possibility of a non-constant entropy.

However, whether or not such a terminal state with a "stationary" contact zone actually can result from the process of penetration is a question to be decided by a complete analysis of the gas-dynamical differential equations for the process of penetration. Our assumption of a stationary contact zone will be valid only approximately. A complete investigation is planned. So far, it has been proved for a weak incoming R-wave that in the first approximation the ensuing contact zone is actually stationary, while it widens like a rarefaction zone in the second approximation.

The situation as described here can be easily understood from the (x,t)-diagram. In symbols the process is described by

$$\underset{\rightarrow}{S} \quad \underset{\leftarrow}{R} \quad \text{———} \quad \underset{\leftarrow}{R} \quad T\,T \quad \underset{\rightarrow}{S}$$

(T T being the symbol for contact zones.)

6. Interaction Between Waves and Contact Surfaces.

There are four cases of such interactions according to whether a shock or a rarefaction wave enters a zone with greater or lower density. In three of these cases our procedure yields immediately a result which is likely to represent the actual phenomena. In the fourth case a more complicated process of penetration must be assumed.

Cases 6 A: $\overset{S}{\rightarrow}$ T $<$ and 6 B: $\overset{S}{\rightarrow}$ T $>$.

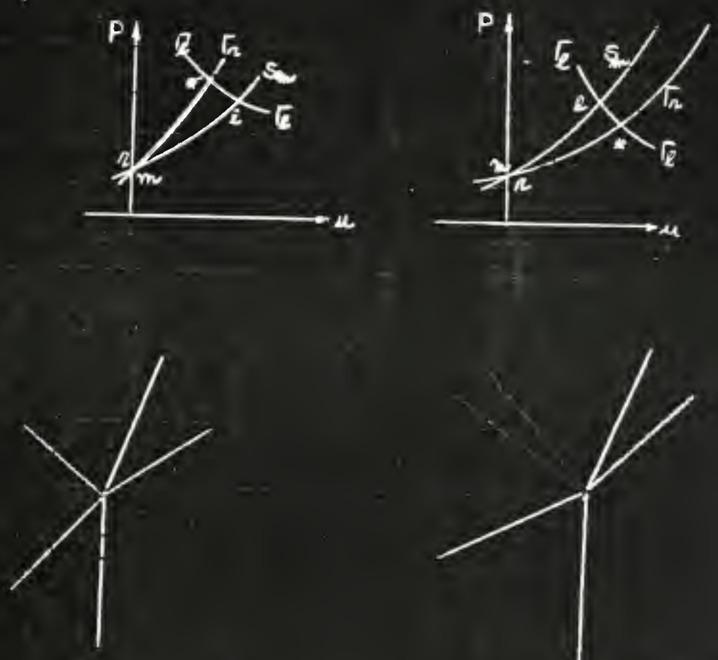
can be considered jointly. In the (u,p)-diagram the points m and r coincide, but the curves S_m and Γ_r connecting this point with the point 1 and m_s respectively differ from each other. As seen from Remark 1, the curve Γ_r passes to the left or the right of the point 1 according to whether $\rho_r > \rho_m$ or $\rho_r < \rho_m$ respectively. The point m_s in which Γ_1 and Γ_r intersect is accordingly on the S-branch in case (6.1), and on the R-branch in case (6.3), i.e., in case 6 A the transition from 1 to m_s is a "reflected" backward shock wave, while in case 6 B a rarefaction wave is reflected. In both cases the

transition from r to m_* is a "transmitted" forward shock wave. The situation is described by the formulae

$$\underset{\rightarrow}{S}\ T < \rightarrow \underset{\leftarrow}{S}\ T\ \underset{\rightarrow}{S}$$

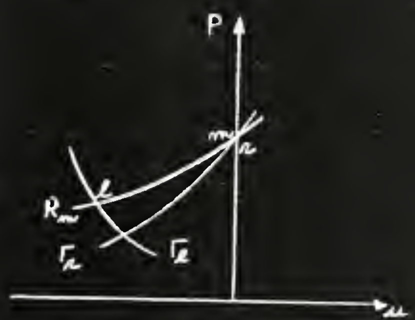
$$\underset{\rightarrow}{S}\ T > \rightarrow \underset{\leftarrow}{R}\ T\ \underset{\rightarrow}{S}$$

and as illustrated by the figures.



Just as simple is the case 6C, $\underset{\longrightarrow}{R}$ T $<$. Here the curve Γ_r through the point m = r passes the point l to the right and the point m_* is on the R-branch of Γ_l . Consequently the transition from l to m_* is a reflected rarefaction wave, while, naturally, the transition from r to m_* is a transmitted rarefaction wave. The reflected wave is comparatively weak, at least if the densities ρ_m and ρ_r do not differ too much. Symbolically

$$\underset{\rightarrow}{R} \quad T< \;\longrightarrow\; \underset{\leftarrow}{R} \quad T \quad \underset{\rightarrow}{R}$$



In case 6 D of a rarefaction wave $\overset{R}{\rightarrow}$ entering a region with a lower density, the discussion on the basis of the (u,p)-diagram yields correspondingly that the curve Γ_r through m = r passes the point 1 to the left and then the point m_s will be on the S-branch of Γ_1. This would imply that the transition of 1 to m_s is effected by a backward shock wave while a rarefaction wave is transmitted into the region r. However, it seems quite unlikely that a reflected shock should develop immediately. It is rather to be expected that an inverted rarefaction wave, i.e., a contraction wave is reflected (cf. pp 6, 15) The new middle state m_{s*} between the forward rarefaction wave and the backward contraction wave is represented by the intersection of the curve Γ_r not with the curve Γ_1 but with the upper branch of the Ψ-curve through 1, which so far was disregarded. (cf. p. 16).

It is clear that the contraction wave will eventually result in a shock. In the process of the development of this terminal shock another forward rarefaction wave and a contact zone will arise. When the shock has become steady a region m_{s**} between this shock and the second rarefaction wave results which according to our assumption is characterized

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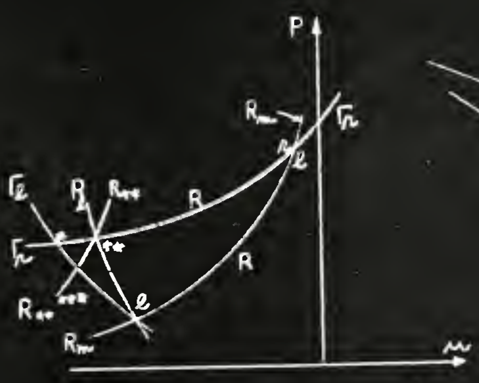
by constant pressure and velocity. This state m_{***} corresponds to the intersection of Γ_l and that curve $\Gamma_{***} = R_{**}$ through the point m_{**} which is determined by the entropy on the left-hand side. The new rarefaction wave will enter the region to the right and lead to a reflected contraction wave in a similar way as before. A sequence of such interactions will ensue which appears to become weaker and weaker: it may be surmised that they will assymptotically approach the terminal state m_* which would have been obtained by the simple original procedure by intersecting Γ_r and Γ_l.



Again it should be stated that our assumption leading to a stationary contact zone is doubtful as in case 5C, and that only a more detailed analysis of the gas-dynamical differential equations can decide the exact or approximate validity of this assumption. However, such an analysis seems rather difficult, and our discussion of the case 6B must altogether be considered as an argument of approximate plausibility.

7. Overtaking of Waves.

Of greater importance than transmission and reflection of waves on contact surfaces is the problem arising when two shock or rarefaction waves facing in the same direction follow each other. When observed from the intermediate zone the leading wave travels with sonic or subsonic velocity, the follow-up wave with sonic or supersonic velocity. Consequently the follow-up wave will overtake the leading wave unless both are rarefaction waves, which remain separated by a zone of constant length.(*)

7 A. Overtaking of Two Shock Waves.

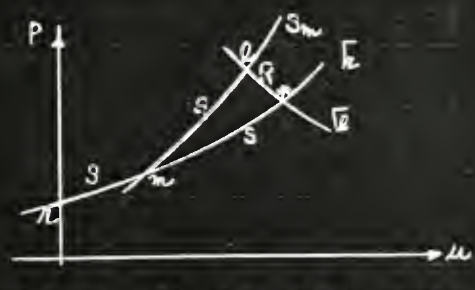
First we consider the case that two forward shocks follow each other. After collision always one

(*) Incidentally, it follows from these remarks that no discontinuity can resolve itself into more than two steady waves. For, should three waves occur, then at least two of them would face in the same direction and thus would overtake each other.

forward shock wave will result. However, the reflected backward wave may be a shock or a rarefaction wave. As von Neumann has already observed it will always be a rarefaction wave if $\gamma \leq 5/3$, or $\mu \geq 4$, which is the case for air but not for monatomic gases. If $\gamma > 5/3$, a shock will be reflected for certain initial pressure ratios. In all cases the new middle zone m_* will be divided by a contact surface.

To establish these statements we again consider the (u,p)-diagram. It is clear that the assumed situations of two forward shocks implies the inequalities,

$$p_r < p_m < p_l, \qquad u_r < u_m < u_l .$$



The point m is evidently on the transition curve Γ_r through r while l is on the $\mathcal{S}$-curve through the

point m. To construct the point m_s we must intersect the left transition curve Γ_1 through 1 with Γ_r. Evidently the point m_s is on the S-branch of Γ_r; this means there will be a transmitted forward shock between r and m_s, and, for that matter, a stronger shock than the original shock between r and m. To decide whether the backward wave is a shock or a rarefaction wave we investigate on which side of the point 1 the curve Γ_r passes. As will be shown in Appendix A3, for $\gamma \leq 5/3$ the curve Γ_r always passes to the right of 1; therefore, in this case, a rarefaction wave is always reflected irrespective of the initial pressure ratios. Symbolically

$$\underset{\rightarrow}{S} \quad \underset{\rightarrow}{S} \rightarrow \underset{\leftarrow}{R} \quad T \quad \underset{\rightarrow}{S} \qquad \text{if } \gamma \leq 5/3$$

$$\rightarrow \left\{ \begin{array}{l} \underset{\leftarrow}{R} \quad T \quad \underset{\rightarrow}{S} \\ \text{or} \\ \underset{\leftarrow}{S} \quad T \quad \underset{\rightarrow}{S} \end{array} \right. \qquad \text{if } \gamma > 5/3$$

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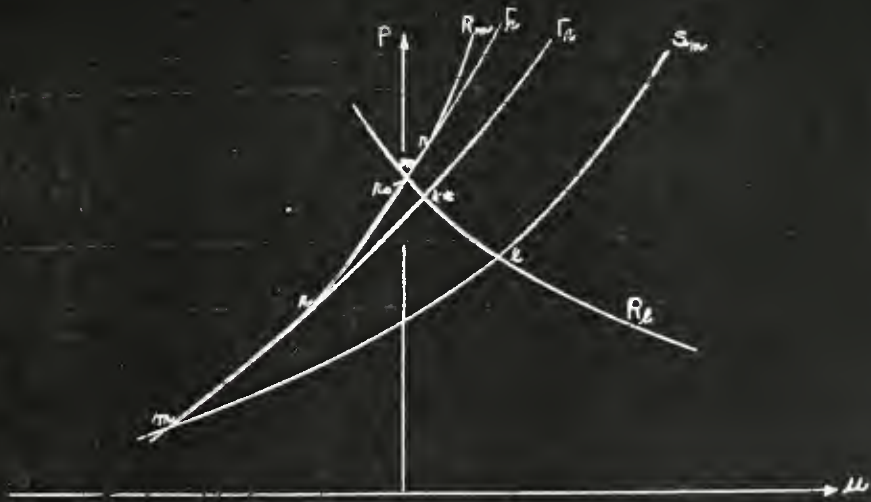
7B. Rarefaction Overtaken by Shock Wave.

Next we consider the phenomenon of a rarefaction wave followed by a shock wave in symbols: $\underset{\rightarrow}{S}\ \underset{\rightarrow}{R}$. In the (u,p)-diagram the point m is on the lower or R-branch of the curve Γ_r through r ; the point 1 is on the upper branch of the $\underset{\rightarrow}{S}$ curve through m. The piece (m,r) of the curve Γ_r is at the same time a section of the $\underset{\rightarrow}{R}$ curve through m . As shown in the appendix, the upper part of the R-curve through a point lies above the upper part of the S-curve through this point, provided $\gamma \leq 5/3$, which we shall assume. This leads to the conclusion that Γ_r lies above the $\underset{\rightarrow}{S}$-curve through m . The point m_* on Γ_r will be on the R- or S-branch of Γ_r depending on the position of the point 1 on $\underset{\rightarrow}{S_m}$, i.e., on the strength of the shock (m,1). The situation becomes somewhat clearer if with a fixed shock (m,1) we consider for the rarefaction wave cases of different intensities. This means that on the (u,p)-diagram the points m and 1 on $\underset{\rightarrow}{S_m}$ are fixed while the point r may be chosen at any point of the curve R_m through m . The curve Γ_1 through the point 1 is also fixed while the curve Γ_r through r depends on the position of the point r , remembering, however, that the R-branch of Γ_r is always a section of $\underset{\rightarrow}{R_m}$. The point m_* will be the

intersection of the curves Γ_1 and Γ_r; it is always on the S-branch of Γ_1. In other words the reflected wave is always a shock. This, however, is only a rough idealization of the terminal situation; actually, as in case 6D, the reflection starts with a contracting wave and leads to a shock only after some time. To obtain this contracting wave we have to pass through 1
the upper branch of the R curve, which intersects Γ_r in r_{**}. Denoting the intersection of R_1 with R_m by r_o, we distinguish the cases where r is below and where r is above r_o on the curve R_m.



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In the first case of a weak rarefaction wave a shock is transmitted, somewhat weakened as is natural. If r coincides with r_c no wave at all is transmitted. The only phenomenon in addition to the reflection is a contact zone. If r is above the point r_c, the point m_{**} always coincides with r_c. This can be interpreted as follows: The process of reflection is exactly as in the intermediate case $(r = r_c)$. That is, one part of the incoming rarefaction waves is compensated by the reflected wave. The remaining part of the incoming rarefaction wave is transmitted without interference through the interaction. Obliterating the finer features of the reflection process, we have symbolically

$$\overset{S}{\rightarrow} \quad \overset{R}{\longrightarrow} \quad \begin{cases} \overset{S}{\leftarrow} \quad T\,T \quad \overset{S}{\rightarrow} \\ \overset{S}{\leftarrow} \quad T\,T \\ \overset{S}{\leftarrow} \quad T\,T \quad \overset{R}{\longrightarrow} \end{cases}$$

Of particular interest is the case where the piston, originally at rest, is retracted at constant speed and then suddenly arrested. A rarefaction wave leaves the piston at the moment when the retraction starts, and a subsequent shock leaves the piston at the moment when it is arrested. The state r then corresponds to the original state of rest, the state m corresponds to the

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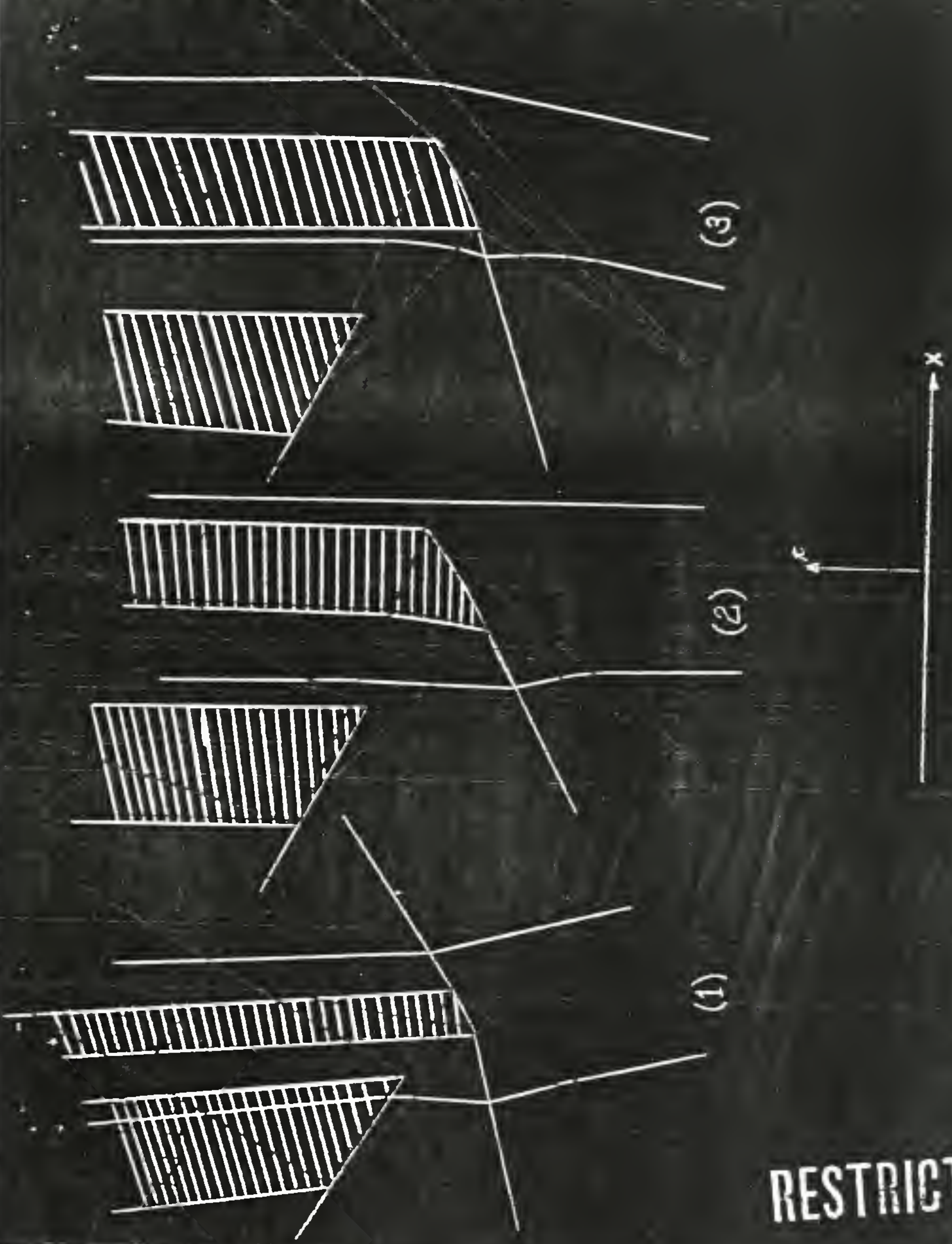
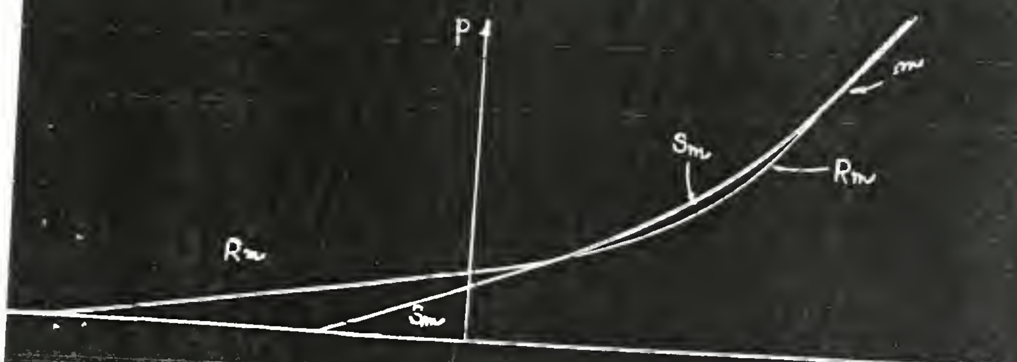


(1)
(2)
(3)
x
r

state adjacent to the piston during the process of retraction, the state l to the state adjacent to the arrested piston. Since $u_r = u_l = 0$, we are obviously in the third of the three cases discussed above where the shock is so weak that it is already compensated by a part of the rarefaction wave.

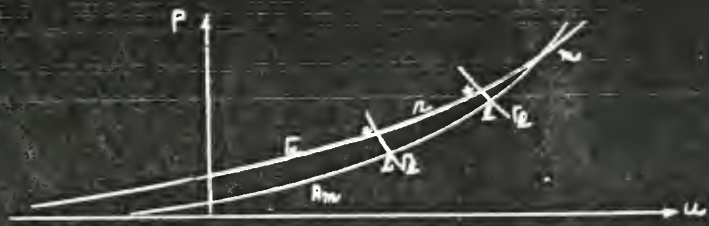
7C. Shock Overtaken by Rarefaction Wave.

In this type of interaction all four combinations of forward and backward waves, just as in Riemann's general problem, may result. In any given case the method of the (p,u)-diagram will lead to the decision among these possibilities. From the point m in the diagram we draw the $\overrightarrow{R}$- and $\overrightarrow{S}$-curves which contain the points l and r respectively.



These two curves start at the point m with the same tangent and curvature. The complications of the discussion arise from the fact that they intersect each other again in the lower branches if $\gamma \leq 5/3$, (viz., where p^o/p_m = .27 for γ = 1.4). (See Appendix A4).

Suppose the point r lies on S_m above the intersection (p_r/p_m > .27), then as easily seen, the curve Γ_r remains above $\underset{\rightarrow}{R}_m$ below the point m. Thus it is clear that, depending on the position of l on R_m, the point m_o is on the S-or R-branch of Γ_r. As a result a weak shock is reflected while the transmitted wave is a shock or rarefaction wave.



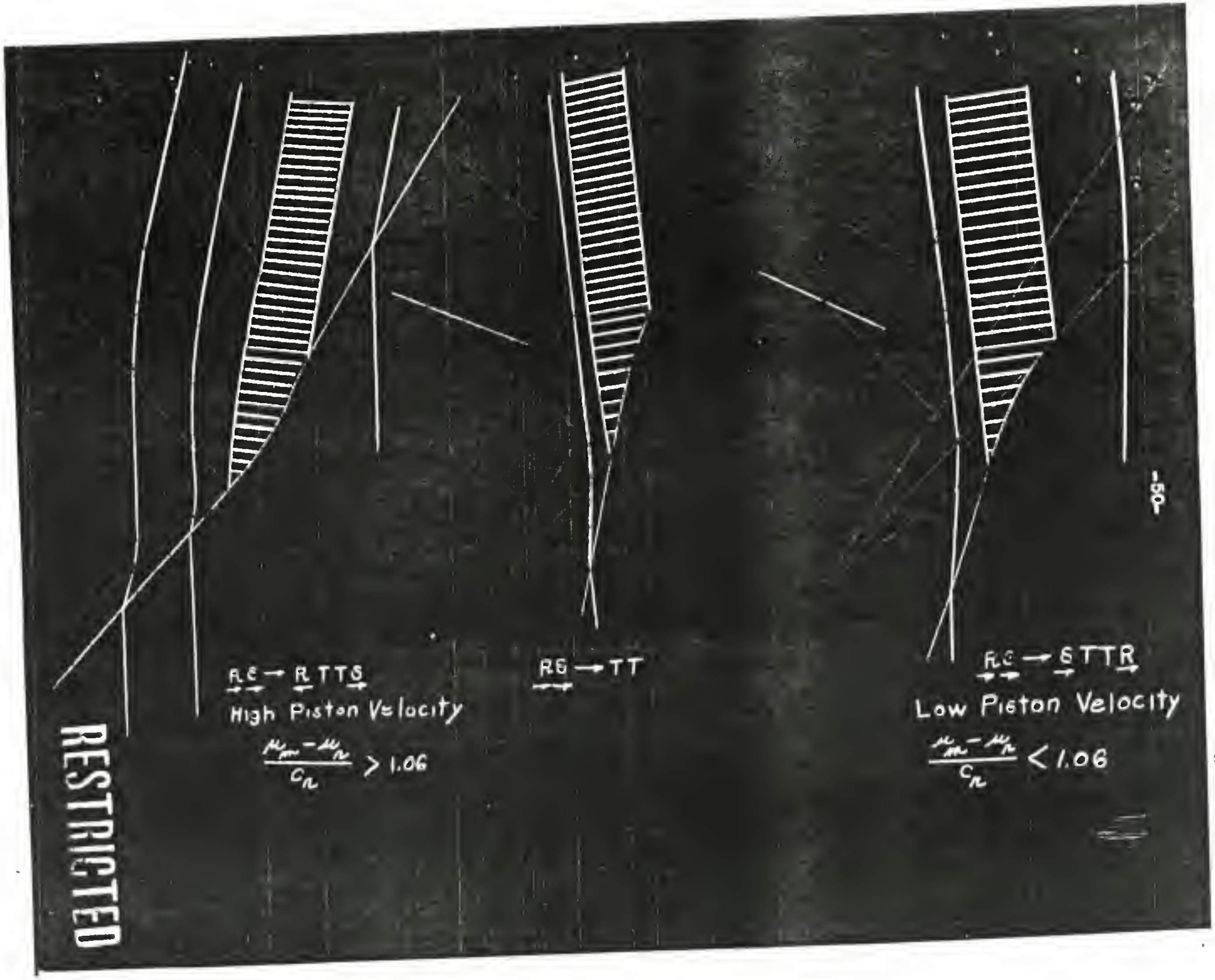
Again, a reflected shock must be understood, as in previous discussions, to result from a reflected contraction wave. Since it is possible that the point r could be situated below the $\underset{\rightarrow}{R}$-curve through m, (p_r/p_m < .27), it follows that a rarefaction wave

may be reflected; in such cases the transmitted wave again may be either a shock or a rarefaction wave. Our statement that the reflected wave is a rarefaction wave, however, needs some qualification. The (u,p)-diagram refers only to the wave as a whole, i.e., to the relation between the states at their ends. It is conceivable that such waves contain contracting parts as well as expanding parts (e.g., if the piston is first decelerated, then accelerated, and finally again decelerated). The contracting part of course then would lead to shocks within the wave. A more detailed inspection of the reflection process makes it likely that the reflected "rarefaction" wave in the present cases contain contraction zones near to their head i.e., the end toward which they are directed.

A few more remarks concerning the transmission and reflection should be made. Under special circumstances, as in case 7B, the two original waves compensate each other so that nothing is transmitted; viz. if in the (u,p)-diagram the R_1-curve intersects Γ_r in the point r. Our diagram suggests an even more radical possibility to occur if the points r and l coincide with the intersection of R_m and S_m: the two waves extinguish each other completely so that neither a transmitted nor

a reflected wave results but only a contact zone, with a non-constant density, entropy, and temperature. It is more likely that the actual phenomenon consists rather in a reflected wave containing contracting and expanding parts such that pressure and velocity at both ends of the wave are the same. Nevertheless, we mention numerical values for this singular phenomenon: $p_r = p_l = (.27)p_m$ and $u_m - u_r = u_m - u_l = (1.06)c_r$, where c_r is the speed of sound in the region r. In a tube this event could be produced by first moving the piston with a speed 6% above that of sound into the undisturbed atmosphere and then by suddenly arresting it. It should be stated, however, that this phenomenon will probably elude experimental verification at any rate, for, if $u_r = u_l$, the two points r and l will be close to each other and hence to m_{ss} when the ratio $(u_m - u_r)/c_r$ varies over a wide margin. This appears from the fact that the $\overrightarrow{R}$ and $\overrightarrow{S}$-curves through m nearly coincide along a large interval.

Three cases for $u_r = u_l$ (arrested piston) are shown in an (x,t)-diagram.


-50-
RE → RTTS
High Piston Velocity
$\frac{\mu_m - \mu_n}{c_n} > 1.06$
RS → TT
RS → STTR
Low Piston Velocity
$\frac{\mu_m - \mu_n}{c_n} < 1.06$
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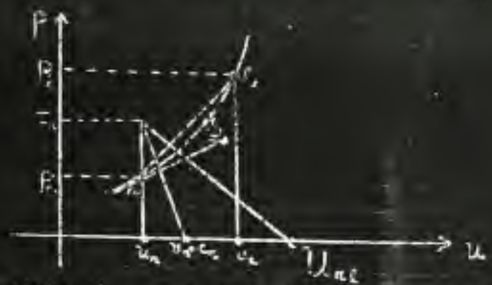
Appendix

A.1. Construction of Sound and Shock Velocities

When a shock transition is represented in a (u,p)-diagram it is possible to obtain the shock velocity and the sound velocities for both sides by a simple geometrical construction.

Let $\underset{\rightarrow}{S}$ be a forward shock connecting a state r with a state 1. We represent these states



by two points (u_r, p_r) and (u_1, p_1) in the (u,p)-diagram, using any lengths $\langle u \rangle$, $\langle p \rangle$ to represent the units of u and p, respectively. In the same diagram we plot the point $(u_r, \widetilde{\tau}_r)$, representing the unit of $\widetilde{\tau}$ by the length $\langle \widetilde{\tau} \rangle = \langle u \rangle^2 / \langle p \rangle$. Through $(u_r, \widetilde{\tau}_r)$ we drop the perpendicular to the chord $(r1)$; its intercept on the u-axis is the shock velocity U_{r1}. This statement follows immediately from relation (2.11) when it is written in the form

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$$(U_{r1} - u_r)/\tau_r = (p_1 - p_r)/(u_1 - u_r) .$$

When the point 1 approaches the point r along the $\overrightarrow{S}$-curve, the shock velocity will approach the sound velocity $u_r + c_r$, where c_r is the sound speed. Accordingly, when we drop through (u_r, τ_r) the perpendicular to the tangent of the $\overrightarrow{S}$-curve at r, we obtain the sound velocity $u_r + c_r$ as the intercept on the u-axis. This statement is also easily verified since

$$\varphi_r'(p_r) = \sqrt{\frac{(\mu - 1)\,\tau_r}{(\mu + 1)\,p_r}} = \tau_r/c_r .$$

In the case of a rarefaction wave the construction of the sound velocity $u_r + c_r$ is identical to that of the preceding case except that the $\overrightarrow{R}$-curve through r is utilized in this case. The verification follows immediately from

$$\psi_r'(p_r) = \sqrt{\frac{(\mu - 1)\,\tau_r}{(\mu + 1)\,p_r}} = \tau_r/c_r .$$

A 2. Head-on Collision of Two Shocks.

In section 5A of this report several unproved statements were made concerning the mathematical frame-work employed in the discussion of the head-on collision of two shock waves. These statements will be proved in the present appendix.

First of all, we shall show that the particle velocity u_* is greater than u_m if the pressure $p_l > p_r$, (5.3). From the relations, cf.(2.10),

$$u_l - u_m = \varphi_m(p_l) \;,\quad u_r - u_l = -\varphi_l(p_r)$$

$$u_* - u_l = -\varphi_l(p_*) \quad u_* - u_r = \varphi_r(p_*) \;,$$

by employing (2.8), we find

$$\text{(A2.1)} \quad u_l - u_r = \varphi_m(p_l) + \varphi_m(p_r) = \varphi_l(p_*) + \varphi_r(p_*)$$

and

$$\text{(A2.2)} \quad u_* - u_m = \tfrac{1}{2}\left[\varphi_*(p_l) - \varphi_*(p_r) + \varphi_m(p_l) - \varphi_m(p_r)\right] .$$

But, in virtue of the monotone character of $\varphi_k(p)$, since it is assumed that $p_l > p_r$, it follows from (A2.2) that $u_* > u_m$.

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Secondly, the relation (5.4), viz., $p_* < p_1 p_r/p_m$, will be proved. We denote by p_c the quantity $p_1 p_r/p_m$. Then, by a simple calculation,

$$\varphi_r(p_c) = \sqrt{\frac{p_r}{p_m}\,\frac{\rho_m}{\rho_r}}\,\varphi_m(p_1) \quad ;$$

since, cf.(2.2),

$$\rho_r/\rho_m = h(p_r/p_m) < p_r/p_m \quad ,$$

whenever $p_r > p_m$, we have $\varphi_r(p_c) > \varphi_m(p_1)$, and similarly $\varphi_1(p_c) > \varphi_m(p_r)$. Consequently,

$$\varphi_r(p_c) + \varphi_1(p_c) > \varphi_m(p_1) + \varphi_m(p_r) \quad ,$$

or, from (A2.1)

$$\text{(A2.3)} \qquad \varphi_r(p_c) + \varphi_1(p_c) > \varphi_r(p_*) + \varphi_1(p_*) \quad .$$

If, now, $p_c \leq p_*$, then, in virtue of the monotonicity of $\varphi_k(p)$, we would have

$$\varphi_r(p_c) + \varphi_1(p_c) \leq \varphi_r(p_*) + \varphi_1(p_*) \quad .$$

The latter statement contradicts (A2.3); hence $p_c > p_*$, in other words,

$$\text{(A2.4)} \qquad p_* < p_1 p_r/p_m \quad .$$

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Finally, in order to investigate the comparative magnitudes of the densities ρ_{l*} and ρ_{r*} in the middle region after the collision, we set

$$\rho_{l*} = \rho_l h(p_*/p_l) \; , \; \rho_{r*} = \rho_r h(p_*/p_r)$$

$$\rho_l = \rho_m h(p_l/p_m) \; , \; \rho_r = \rho_m h(p_r/p_m) \; .$$

Introducing the function

$$\chi(x) = h(p_*/x)\, h(x/p_m) \; ,$$

we have

$$\text{(A2.5)} \qquad \rho_{l*} = \rho_m \chi(p_l) \; , \; \rho_{r*} = \rho_m \chi(p_r) \; .$$

Now, the function $\chi(x)$ has the following simple properties, which result immediately from those of $h(x)$:

$$\chi(x) > 1 \quad \text{for } x > 0 \; ,$$
$$\chi'(x) \gtrless 0 \quad \text{according as } x \lessgtr \sqrt{p_m p_*} \; ,$$
$$\chi(x) = \chi(y) \; , \text{ if and only if } xy = p_m p_* \; ,$$
$$\chi(x) < \chi(y) \; , \text{ if } \begin{cases} x < y \text{ and } xy < p_m p_* \; , \\ \quad\text{or} \\ x > y \text{ and } xy > p_m p_* \; . \end{cases}$$

Since it was assumed that $p_l > p_r$, it follows, in

virtue of (A2.4), that

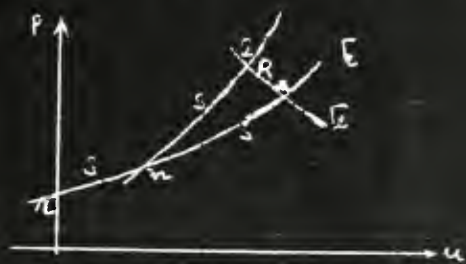
$$\chi(p_1) < \chi(p_m) ,$$

or, from (A2.8), that

$$\rho_{1*} < \rho_{r*}$$

A 3. Overtaking of Two Shock Waves

In the discussion of section 7a, von Neumann's statement that the reflected wave will always be a rarefaction wave if $\gamma \leq 5/3$, or $\mu^2 \geq 4$ was mentioned. A short proof will be given here.



Consider a point m on the $\underrightarrow{S}$-curve through a point r in the (u,p) plane. All that need be proved is that the continued $\underrightarrow{S}$-curve through m remains above the $\underrightarrow{S}$-curve through r. We may express this statement as follows:

(A3.1) $\varphi_r(p) > \varphi_r(p_m) + \varphi_m(p)$, for $p > p_m > p_r > 0$

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Letting $x = p/p_m$ and $y = p_r/p_m$, then, according to (2.2), we may express $\tau_r = \frac{1}{\rho_r}$, occurring in $\varphi_r(p)$, as

$$\tau_r = \tau_m \frac{y + \mu}{\mu y + 1} .$$

By the introduction of the function

$$g(x,y) = (x-y)\sqrt{\frac{1}{\mu x + y} \cdot \frac{y + \mu}{\mu y + 1}} ,$$

we have

$$\varphi_r(p) = \sqrt{\frac{(\mu - 1)\tau_m}{p_m}} \quad g(x,y) ,$$

$$\varphi_m(p) = \sqrt{\frac{(\mu - 1)\tau_m}{p_m}} \quad g(x,1) .$$

Hence, the statement (A3.1) can be written

$$g(x,y) > g(1,y) + g(x,1) \quad ;$$

or, since $g(1,1) = 0$, we need only prove the following statement equivalent to (A3.1) :

(A3.2) $g(x,1) - g(x,y) - g(1,1) + g(1,y) < 0$,

for $0 < y < 1 < x$.

Obviously, in order to demonstrate (A3.2) it is sufficient to show that

(A3.3) $\frac{\partial^2 g}{\partial x \partial y} < 0$, for $0 < y < 1 < x$.

We find that

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\mu}{2} \frac{\left[(1+2\mu^2)\mu x-(2+\mu)y\right](\mu y+1)(\mu+y)-(\mu x+y)\left[\mu x+(2+\mu^2)y\right](\mu^2-1)}{(\mu x+y)^{3/2}\,(\mu+y)^{1/2}\,(\mu y+1)^{3/2}}$$

The numerator can be regrouped in the following form:

$$\begin{aligned}
&-\mu(\mu^2+1)(\mu-4)(x-y)\,y-\mu^2(\mu^2-1)(x-1)^2\\
&-\mu^2(2\mu^2-2\mu-3)(x-1)\,(x-y)\\
&-\mu^2(\mu^2-2\mu-2)(y-1)^2\\
&-\mu(\mu+2)(y-1)^2\,y-\mu^2(2\mu+1)(x-1)(y-1)^2
\end{aligned}$$

This expression is obviously negative for $0 < y < 1 < x$, if $\mu \geq 4$. Thus von Neumann's statement is proved.

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A 4. Overtaking of Shock and Rarefaction Waves.

For the discussion of the overtaking of a shock wave by a rarefaction wave in sections 7C and vice-versa in 7B it was necessary to know the relative positions of the R- and S-curves through the same point in the (u,p) plane; i.e., to know whether or not $\varphi_k(p) < \psi_k(p)$.

With the notations $x = p/p_k$,

$$\text{(A4.1)}\qquad \begin{cases} \Phi(x) = (1+\mu x)^{-\frac{1}{2}}(x-1), \\ \Psi(x) = (1+\mu)^{\frac{1}{2}}\left(x^{1/(1+\mu)} - 1\right), \end{cases}$$

we have

$$\varphi_k(p) = \sqrt{(\mu-1)\,p_k\,\tau_k}\;\Phi(p/p_k)$$

$$\psi_k(p) = \sqrt{(\mu-1)\,p_k\,\tau_k}\;\Psi(p/p_k)\ .$$

Differentiating (A4.1), we find

$$\Phi'(x) = \tfrac{1}{2}(2+\mu+\mu x)(1+\mu x)^{-3/2}\ ,\quad \Phi'(1) = (1+\mu)^{-\frac{1}{2}}$$

$$\Psi'(x) = (1+\mu)^{-\frac{1}{2}}\,x^{-\mu/(1+\mu)}\ ,\quad \Psi'(1) = (1+\mu)^{-\frac{1}{2}}$$

$$\Phi''(x) = -\tfrac{1}{4}\mu(4+3\mu+\mu x)(1+\mu x)^{-5/2}, \quad \Phi''(1) = -\mu(1+\mu)^{-3/2},$$

$$\Psi''(x) = -\mu(1+\mu)^{-3/2}\, x^{-(1+2\mu)/(1+\mu)}, \quad \Psi''(1) = -\mu(1+\mu)^{-3/2},$$

$$\Phi'''(x) = \tfrac{3}{8}\mu^2(6+5\mu+\mu x)(1+\mu x)^{-7/2}, \quad \Phi'''(1) = \tfrac{9}{4}\mu^2(1+\mu)^{-5/2}$$

$$\Psi'''(x) = \mu(1+2\mu)(1+\mu)^{-5/2}\, x^{-(2+3\mu)/(1+\mu)}, \quad \Psi'''(1) = \mu(1+2\mu)(1+\mu)^{-5/2}$$

If $\mu > 4$, then $\frac{9}{4}\mu > 2\mu + 1$ and hence $\Phi'''(1) > \Psi'''(1)$. Thus, if $0 < x - 1$ is sufficiently small, $\Phi(x) < \Psi(x)$. However, $\Phi(0) = -1$ and $\Psi(0) = -(1+\mu)^{\frac{1}{2}}$, so that for $\mu > 4$ $\Phi(0) > \Psi(0)$. Hence the functions $\Phi(x)$ and $\Psi(x)$, and therefore the curves $u = \varphi_k(p)$ and $u = \psi_k(p)$ intersect between $x = 0$ and $x = 1$, or $p = 0$ and $p = p_k$ respectively. By calculation it was found for $\gamma = 1.4$ that this happens near $x = .27$. This is the result that was used in section 7C.

In section 7B use was made of the fact that $\varphi_k(p) > \psi_k(p)$ for $p > p_k$. This is equiv to

$$\Phi(x) > \Psi(x) \qquad \text{for } x > 1.$$

For $x > 1$ and $\mu > 4$, we have

$$\frac{6 + 5\mu + \mu x}{1 + \mu x} > 6 \; , \quad \frac{1 + \mu}{1 + \mu x} > \frac{1}{x} \quad ;$$

hence $\Phi'''(x) \geq x^{-5/2} \; \Phi'''(1)$. Furthermore, since $\frac{2 + 3\mu}{1 + \mu} > \frac{5}{2}$

$$\Psi'''(x) \leq x^{-5/2} \; \Psi'''(1) \text{, so that}$$

$$\Psi'''(x) \leq \Phi'''(x) \text{. Consequently } \Psi(x) < \Phi(x)$$

for $x > 1, \mu > 4$, i.e., the φ- and ψ-curves do not intersect for $x > 1, \mu > 4$.

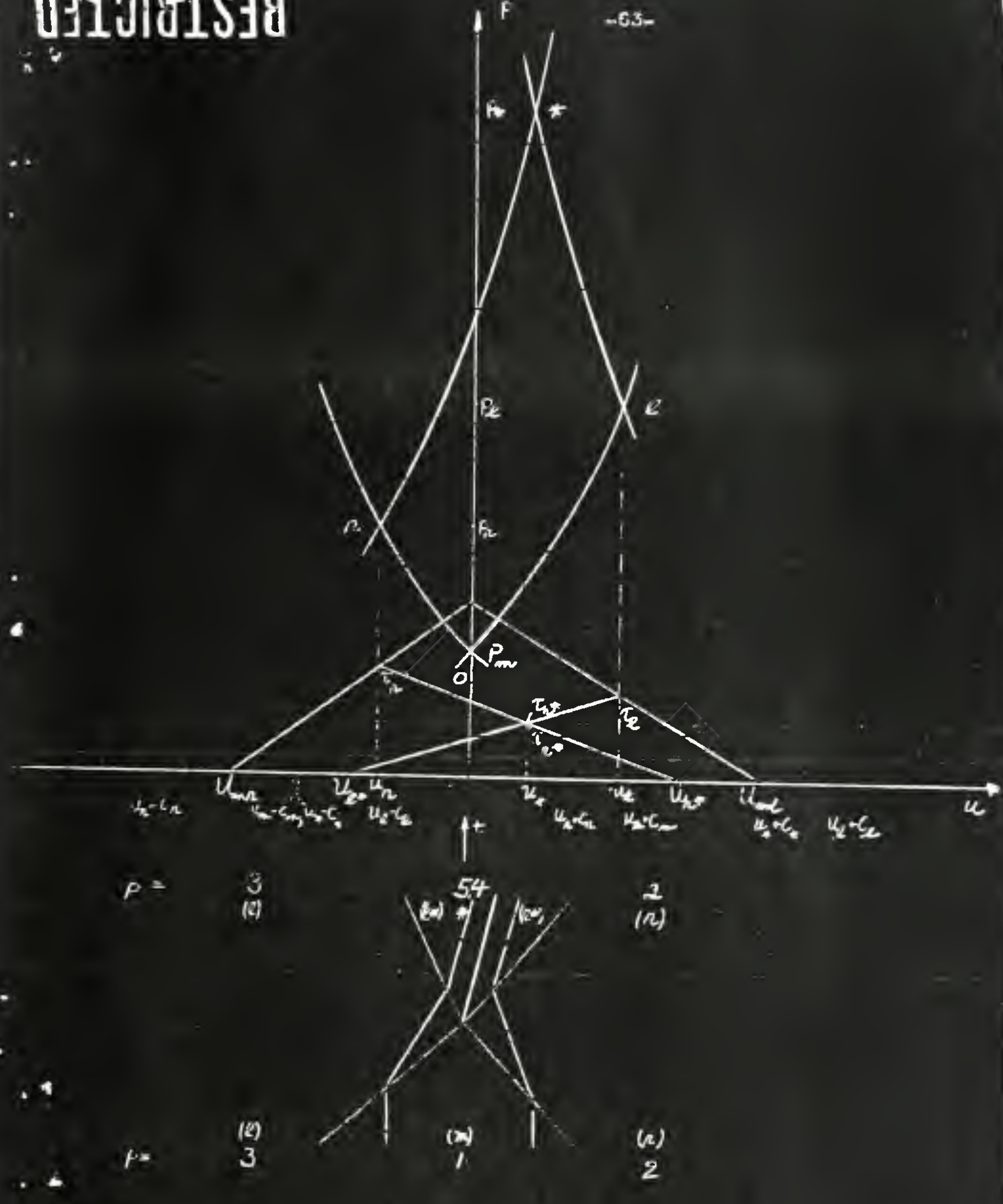
A 5. <u>An Example of the Head-on Collision of Two Shocks.</u>

In this section we present graphically and numerically a special example of the head-on collision of two shocks. The pressure ratios of the colliding shocks are assumed to be $p_l/p_m = 3$ and $p_r/p_m = 2$; the pressure in the resulting middle region is then found to be $p_* = 5.37\ p_m$.

The figure on the next page is self-explanatory; the various particle, shock, and sound velocities indicated in the figure were found by the graphical construction explained in Appendix 1. The results of a numerical calculation are tabulated below:

$p_r/p_m = 2.00$	$(u_m - u_r)/c_m = 0.516$
$p_l/p_m = 3.00$	$(u_l - u_m)/c_m = 0.858$
$p_*/p_m = 5.37$	$(u_* - u_m)/c_m = 0.336$

$\rho_r/\rho_m = 1.62$	$(u_m - U_{mr})/c_m = 1.37$	$c_r/c_m = 1.10$
$\rho_l/\rho_m = 2.11$	$(U_{ml} - u_m)/c_m = 1.65$	$c_l/c_m = 1.18$
$\rho_{r*}/\rho_m = 3.20$	$(U_{r*} - u_m)/c_m = 1.22$	$c_{r*}/c_m =$ [illegible]
$\rho_{l*}/\rho_m = 3.18$	$(u_m - U_{l*})/c_m = 0.69$	$c_{l*}/c_m = 1.44$


F
P_m
O
u
p = 3 54 2
(l) (r)
(l) (m) (r)
p = 3 1 2

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Courant, Richard
Interaction of shock and rarefaction waves in one-dimensional media.